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On the geometry of homogeneous turbulence, with stress on the fractal dimension of the iso-surfaces of scalars

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This paper studies several geometric aspects of the Poisson and Gaussian random fields approximating Burgers k^{-2} and Kolmogorov $k^{-\frac{5}{3}}$ homogeneous turbulence. In particular, simulated sample scalar iso-surfaces (e.g. surfaces of constant temperature or concentration) are exhibited, and their relative degrees of wiggliness are shown to be best characterized by saying that the corresponding fractal dimensions are respectively equal to $3-\frac{1}{2}$ and $3-\frac{1}{3}$.

1. Introduction

Turbulence in fluids raises a variety of interesting and practically important problems of geometry, which have not, so far, received the full attention they deserve. The theory of stochastic processes (much influenced, through N. Wiener, by Perrin's (1913) work on Brownian motion and G. I. Taylor's early papers on turbulence) has grasped fully the peculiar and 'pathological' shapes of randomly generated lines, and either borrowed or developed analytic and geometric tools to describe this kind of irregularity. But geometry (in contrast to analysis) has hardly at all been applied to the specific random surfaces of turbulence. This failure is particularly surprising because turbulent shapes are readily visualized and therefore almost cry out for proper geometrical description. The present paper is one of a series I am in the process of devoting to this goal. Two which have appeared (Mandelbrot 1972, 1974) were concerned with intermittency. The present work, however, which can be read independently, returns to a more traditional context, and investigates certain geometric aspects of the random fields of the classic Kolmogorov 1940 theory of homogeneous turbulence. More precisely, we shall study two approximations. The more familiar is the zero-mean random Gaussian field the variance of whose increments obeys the $\frac{2}{3}$ law. The less familiar approximation, to be called a Poisson field, can also be viewed as a (new) algorithm designed to make it possible to simulate the above Gaussian field on a computer. The elementary steps of this algorithm may seem to have a possible concrete interpretation in terms of 'shocks', but will in fact turn out to be mere mathematical devices. To give an idea of the geometry, we shall exhibit some views of actual simulations of the iso-surfaces of scalar quantities, such as the surfaces of constant temperature, and shall comment upon their shapes. We shall stress the fact that in the case of Gaussian fields the iso-surfaces are convoluted to such an extraordinary extent that it is best to consider them as lying 'in-between' ordinary surfaces and solids, and more precisely, as having a dimension equal to $\frac{8}{3}$. Similarly, although in this case the proof is as yet incomplete, it appears that the four-dimensional Euclidean graph of the function giving the temperature at a point has a dimension equal to $\frac{11}{3}$. The concept of a fractional (Hausdorff) dimension has been known in pure mathematics for over half a century, a good reference being Rogers (1970). It is particularly potent in describing the fine-structure of random functions. However, it remained a little-known curiosity, even among mathematicians, and no presentation directed towards scientists was attempted until recently (Mandelbrot 1975*c*), presumably because no concrete application was suspected until it was injected systematically into natural science, first through the study of certain noises (Mandelbrot 1965), then through the shape of the earth's surface (Mandelbrot 1967) and ultimately through the study of turbulence (Mandelbrot 1972, 1974), where its importance is far from having been exhausted. For reasons I cannot describe here, it is preferable to replace the term 'fractional dimension' by 'fractal dimension'; the corresponding sets can be called fractals. Different applications of fractals to turbulence are described in Mandelbrot (1975*d*) and Scheffer (1975), and in Mandelbrot (1976).

In the hope of making the structure of the whole argument clearer, it will be carried out first for the corresponding Poisson and Gaussian approximations to Burgers homogeneous 'turbulence'; though this is extremely crude as a model, it will be so much simpler that it will provide useful insight into reality. For it, the fractal dimension of the iso-surfaces is 2·5.

The discussion will include an application of another concept which used to be considered a mathematical curiosity. A scalar Burgers Gaussian field turns out to be identical with Lévy's (1948) independently developed concept of a 'Brownian function' in space; by analogy, we shall propose for the scalar Kolmogorov Gaussian field the term 'fractional Brownian function'.

2. The geometry of random scalar fields with Burgers variance and with Poisson and Gaussian distributions (Brownian functions of a point)

In the one-dimensional Burgers model, turbulence has a spectral density proportional to k^{-2}. At least since von Neumann (1963), it has been customary to apply the term 'Burgers turbulence' also to a collection of step-like discontinuities in three dimensions. In the case we shall study first, that of a scalar turbulence field, say the temperature $B(P)$ at the point P or the concentration of an inert contaminant (see Corrsin 1951), this field is such that, given any two points P' and P'', $\langle[B(P')-B(P'')]\rangle = 0$ and $\langle[B(P')-B(P'')]^2\rangle = |\overline{P'P''}|$.

2.1. *Poisson fields*

A precise mathematical model of a Burgers field is the Poisson field, one that results from an infinite collection of 'steps' (say steps of temperature) whose directions, locations and intensities are given by three infinite sequences of mutually independent random variables. The locations can be determined from the distances from the planes carrying the steps to the origin O, and they must form a Poisson sequence of positive numbers R_n, which is by definition such that the probability of finding one between $R \geqslant 0$ and $R+dR \geqslant 0$ is dR/μ, with $\mu > 0$. The directions of these planes can be determined by those of the altitudes drawn to them from O, and they must be given by a sequence of points H_n on the unit sphere such that the probability of finding one in any domain of area dS is $dS/4\pi$. Finally, the amplitudes can be represented by a sequence of random quantities Q_n that are arbitrary except that they have a symmetric distribution and a finite variance $\langle Q_n^2 \rangle$, which will be assumed to be normalized to unity. Because $\langle Q_n^2 \rangle = 1$, $\langle Q_n \rangle$ is finite, and because of symmetry it must vanish. Thus to each n there will correspond, first, the point V_n such that $\overline{OV_n} R_n = \overline{OH_n}$, second, the plane perpendicular to $\overline{OV_n}$ through V_n, defined as the locus of points P such that $\overline{OP} . \overline{OH_n} = R_n$, and finally, the function $D_n F(P)$ that vanishes where $\overline{OP} . \overline{OH_n} < R$ (in particular, at the point O), equals Q_n where $\overline{OP} . \overline{OH_n} > R_n$ and equals $\frac{1}{2}Q_n$ where $\overline{OP} . \overline{OH_n} = R_n$. Because of this last property and the symmetry of the distribution of Q, the distribution of $D_n F(P)$ is isotropic. Note also that, even if Q is Gaussian, the joint distribution of two or more values of $D_n F(P)$ is not Gaussian; hence $D_n F(P)$ is never a Gaussian field. Adding all the contributions $D_n F(P)$, one defines a Poisson field as

$$F(P) = \sum_{n=1}^{\infty} D_n F(P).$$

The above construction involves a specific origin O, but it is easy to see that the distribution of the planes of discontinuity is invariant as O is moved around. The same is therefore true of the field $F(P)$, in the sense that the distribution of $F(P') - F(P'')$ for any two fixed points P' and P'' is invariant. In particular, the number of planes of discontinuity intersecting any bounded domain is almost surely finite. When this domain is an arbitrary segment $P'P''$, the number of intersecting planes is a Poisson random variable of expectation $\lambda|\overline{P'P''}|$, where λ is proportional to μ; thus one has $\langle F(P') - F(P'') \rangle = 0$ and

$$\langle [F(P') - F(P'')]^2 \rangle = \lambda |\overline{P'P''}|,$$

as Burgers wanted.

The fact that $F(P') - F(P'')$ is only affected by 'local' planes, defined as those which intersect $\overline{P'P''}$, expresses that, in one sense at least, this field is local; but the matter of local versus global properties also has other aspects, to which we shall return in §4.

The probability distribution of a scalar Poisson field depends on the distribution of Q; as a result, it is not universal. For example, unless Q itself is Gaussian, the probability distribution of $F(P) - F(O)$ depends strongly upon the number of contributing terms $D_n F(P)$, and hence upon the value of $|\overline{OP}|$. For large $|\overline{OP}|$, the central limit theorem applies, and $F(P) - F(O)$ tends to a Gaussian distribution irrespective of the distribution of Q. For small $|\overline{OP}|$, on the contrary, $F(P)$ may have a variety of shapes.

2.2. *Brownian fields*

Von Neumann (1963, p. 450) asserted that "It would appear [that the Burgers approach] describes a fixed number of shocks of fixed size correctly, but it seems questionable whether its conclusions still apply to an asymptotically (with time) increasing number of (individually) asymptotically weakening shocks. Yet, this is probably the pattern of hydrodynamical shocks in those cases where they combine with turbulent motion." It is not clear which "conclusions" the author had in mind, but the mathematical construction of a field made of such shocks is not only possible, but easy. Indeed, shortly before von Neumann wrote those lines, Lévy (1948) had defined the scalar Brownian function $B(P)$ of a point P as the scalar Gaussian field having the characteristic properties that $\langle B(P')-B(P'')\rangle = 0$ and $\langle |B(P')-B(P'')|^2\rangle = |\overline{P'P''}|$. First of all, this field is clearly the (unique) Gaussian interpolate of a Burgers field. In addition, by making λ into a parameter and denoting the Poisson field by $F_\lambda(P)$, $B(P)$ can be shown to be the $\lambda \to \infty$ limit of the infinite sequence of Poisson fields $\lambda^{-\frac{1}{2}}F_\lambda(P)$. This limit process implements fully the notion of an increasing number of increasingly weak discontinuities. (The concept of the limit of a random field has many different aspects; it suffices there that, for any set of points P_n, the probability distribution of the vector of co-ordinates $\lambda^{-\frac{1}{2}}F_\lambda(P_n)$ should converge to that of the vector of co-ordinates $B(P_n)$.) Conversely, the possibility of defining $B(P)$ through this limit process eliminates the artificiality (underlined by Lévy) that has characterized earlier methods for generating $B(P)$, and it yields a method (not known to Lévy) for performing computer simulations.

A Brownian field is extremely irregular, but (given the impossibility of four-dimensional graphics) this fact cannot be illustrated directly. We may, however, study its planar sections. The simplest among them are (anyhow) of independent interest, since Mandelbrot (1975*a*, *b*) proposed the function $B(x, y, 0)$ as a crude model of the earth's surface. The rectilinear cross-sections are even simpler, since along the x axis, to take an example, $B(P)$ reduces to the ordinary Brownian motion $B(x, 0, 0)$. Even more illustrative of $B(P)$, however, is the structure of its iso-sets $B(P) = \text{constant}$, as it is fully exemplified by the set $B(P) = 0$.

2.3. *Poisson iso-surfaces*

For a continuous function $G(P)$, an iso-surface is a set of points where G takes the same value, but this concept does not apply to $F(P)$, because it is not continuous. It follows, for example, that the points where $F(P) = 0$ reduce almost surely to a small volume around O bordered by a finite number of planes of discontinuity. (However, one may well choose to extend the definition of the iso-set $F(P) = 0$ to include all the points which have a neighbourhood where $F \geqslant 0$ and another neighbourhood where $F \leqslant 0$. If one does so, the iso-set almost surely includes a surface composed of an infinite number of small bounded pieces of plane; in addition, it includes the above small volume near O. On the other hand, if the constant C is chosen at random with a continuous distribution, the iso-set $F(P) = C$ is simply a surface.)

FIGURE 1. Four successive slices of a computer-generated approximation to the volume $B(P) > 0$, where $B(P)$ is the Gaussian approximation to a Burgers scalar field at a point P in space. Clearly, these drawings are much too smooth to represent actual turbulence (see figure 2), but it has not escaped the author's attention that they are reminiscent of various portions of the earth's surface: Greece, the Sea of Okhotsk (as seen in a mirror), the Gulf of Siam or perhaps Western Scotland. In other publications, the author has shown that, while such a resemblance is in no way unusual, many other parts of the world are smoother, and require a fractional Brownian model with a larger value of H than either of the two which occur in turbulence. Note that, although the black and white regions are identical in their statistical properties, the white one is by far the larger. The above graphs do not include such 'empty' portions of the square slices.

2.4. *Cut-offs*

When one takes account of viscosity, the discontinuous functions $D_n(P)$, the Poisson field and the limiting Brownian field $B(P)$ have no physical meaning. Nevertheless, the fine-structure of $B(P)$ is very interesting, and even more so the fine-structure of the Gaussian field $B^*(P)$ with Kolmogorov variance (see § 3). We hope, of course, that this fine-structure is not entirely due to the use of the Gaussian approximation, but (being unable to tell) we choose not to worry about this. Even in cases where the fine-structure corresponds to inaccessible asymptotics, there is a good practical reason for studying it: it simplifies and clarifies the fine-structures above the cut-off. Such was the opinion of Perrin (whom we shall soon quote on a related matter) concerning the importance of the fine-structure of Brownian movement.

In addition, the Brownian field $B(P)$ has an infinite external scale, which may be physically unrealistic, but is very difficult to change without modifying the model profoundly.

2.5. *Brownian iso-surfaces*

The field $B(P)$ is continuous, so that the set $B(P) = 0$ is a continuous surface and its statistical structure and shape are characteristic of all the iso-surfaces $B(P) =$ constant. Figure 1 shows several successive 'slices' of an approximation to the set $B(P) = 0$, constructed on the basis of a special Poisson field $F(P)$, such that Q_n follows an approximation to a Gaussian distribution. The values of $B(P)$ were first evaluated over a spatial grid of 51^3 points; then, because no standard computer routine was available for spatial interpolation, smooth interpolates of the lines where $B(P) = 0$ were drawn separately for each horizontal planar grid of 51^2 points. Hence the smoothness of the line drawn in each of these planes is an artifact, and so are the discontinuities seen between the different planes. The main feature of the limit field $B(P)$ is that it is isotropic and has no intrinsic scale at all; therefore the local detail seen on the surfaces $B(P) = 0$ should be disregarded, and replaced by interpolating mentally some reduced-scale versions of what we see of the medium-sized and large detail. The result is extremely (in fact infinitely) irregular: the biggest visible piece is surrounded by blobs, jetsam and flotsam of all shapes and sizes, 'negative image' versions of which are scattered in its interior; in addition, the outside branches out into filaments of every kind penetrating the inside, and conversely. A special hindrance to intuition is that, the distribution of $B(P)$ being symmetric, the zones where $B(P) > 0$ and $B(P) < 0$, respectively, are statistically identical in their geometric properties. Nevertheless, the notion that a shape can be qualitatively identical to its complement is not as difficult to comprehend as it might seem. This is especially true in the present case. Although the regions $B > 0$ and $B < 0$ have identical expected volumes, the volumes actually observed in a sample (more precisely, those of the intercepts of these regions with a large sphere of centre O) may be expected to be quite different.

Digression. It is interesting, at this point, to quote an excerpt concerning colloids from the preface of a classic book, Perrin (1913).

> Consider one of the white flakes that are obtained by salting a soap solution. At a distance its contour may appear sharply defined, but as soon as we draw nearer its sharpness disappears. The eye no longer succeeds in drawing a tangent at any point on it; a line that at first sight would seem to be satisfactory, appears on closer scrutiny to be perpendicular or oblique to the contour. The use of magnifying glass or microscope leaves us just as uncertain, for every time we increase the magnification we find fresh irregularities appearing, and we never succeed in getting a sharp, smooth impression, such as that given, for example, by a steel ball.

A very reasonable implementation of what Perrin had in mind seems to be provided by the surface $B(P) = 0$. The surface $B^*(P) = 0$ of §4 might be even better. Further, the analogy may well go beyond mere geometry; it may be that, first, Perrin's flakes fill the zones where some threshold of concentration of soap is exceeded, and second, that said concentration is a manifestation of very mature turbulence.

2.6. *Rectilinear cross-sections of the Poisson and Brown iso-surfaces*

As a preliminary, the plane cross-section of $B(P) = 0$ is an iso-line of $B(x, y, 0) = 0$. Thus it is an ocean level line of a crude image of the earth's surface; hence it is a crude image of an ocean coastline (Mandelbrot 1967, 1975*a*). Next, the rectilinear cross-section of $B(P) = 0$ is the set of zeros of ordinary Brownian motion. Similarly, the rectilinear cross-section of $F(P) = 0$ in a Poisson field is the set of zeros of a Poisson process. Both barely differ from the zeros of a random walk, an example of which is exhibited in Feller (1968, chap. III, figure 4) and discussed in his §III.6, to which the reader is referred. Their most striking feature lies in their strongly hierarchical nature: they come in clusters, which combine into super clusters, then in turn into super super clusters, etc., *ad infinitum*. As one views the whole ever more closely, fine detail that seemed like a smooth piece of surface gradually decomposes into many separate folds; correspondingly, what seemed like an isolated intersection of a smooth surface and a line decomposes into many distinct points. As one comes closer to any of the folds, the same process repeats again and again. In the case of a Poisson field, the process is finite, but in the case of a Brownian field, it is endless, ever finer folds being continually revealed. Finally, in a Poisson or Brownian field that has been smoothed to take account of viscosity, a finite number of steps leads to a surface that has neither flat portions nor bends.

2.7. *The set $B(P) = 0$ is fractal and its dimension is 2·5*

Our intuitive feeling for geometric shape has been trained by the study of patterns that are enormously simpler than the ones we are now investigating, such as threads and veils, or, to use the terminology of Kuo & Corrsin (1972), "blobs, rods, slabs and ribbons". We therefore experience great difficulty in comprehending and labelling patterns that are extremely irregular. For example, the set $B(P) = 0$ is, intuitively, 'more space filling' than an ordinary surface or veil, but of course is 'less space filling' than an ordinary solid. Let us now demonstrate that the loose notion of unequal degrees of filling can be made more rigorous, and at the same time can be strengthened, by showing that it can be measured by a single number D, with a value below 3, to be called a 'fractal' (sometimes 'fractional') dimension. This concept (as we have already said) has been featured in several papers concerning the intermittency of turbulence, but the results obtained there are not required in order to follow the present argument. I even believe that acceptance of the concept of D will be promoted if the present discussion abstains from referring to earlier applications. Also, in order to minimize the technical difficulty, we shall adopt an approach that is somewhat unusual and admittedly controversial. Among many near-identical definitions of D, we shall pick one that is both intuitive and conducive to a very direct proof. The equivalence between this definition and others (which are more usual but almost certainly are unknown to the reader) will not be tackled here.

Let us select in an 'appropriate fashion' (see below) a large, test cube in space, whose side L will be the external scale, then let us subdivide it into $(L/\eta)^3$ small cells, whose side η will be the internal scale, and finally, let us count the number $N(L, \eta)$ of cells that include at least one point where $B(P) = 0$. When one proceeds in this fashion with an ordinary curve, one that has a well-defined length, it can happen that the curve and the test cube do not intersect, in which case $N(L, \eta) = 0$; but such cases are without interest. In other words, the appropriate test cubes are those which actually intersect the object being tested, and for them one finds that $N(L, \eta)$ is about L/η (the meaning of 'is about' need not be discussed here). Similarly, if a test cube is chosen such that it intersects an ordinary surface $N(L, \eta)$ is about $(L/\eta)^2$. Finally, with an ordinary solid, the appropriate $N(L, \eta)$ is about $(L/\eta)^3$. In each case, the exponent is simply the

Euclidean number of dimensions. With the pattern of points $B(P) = 0$, however, we shall prove in a moment that $\langle N(L,\eta)\rangle = (L/\eta)^{\frac{5}{2}}$; thus, by analogy with the above role of the Euclidean dimension, one may call $D = 2{\cdot}5$ the dimension of our pattern. It falls below 3 by an amount equal to one-half of the exponent of $|\overline{P'P''}|$ in the variance of $B(P') - B(P'')$. Let us mention in advance that in the Kolmogorov model, to be discussed in §3, the corresponding dimension will follow the same general rule, hence it will be found to be equal to $D = \frac{8}{3} = 3 - \frac{1}{3}$. The amount by which this value exceeds 2 is even greater than that for the present $D = 2{\cdot}5$, and indeed the pattern is even more extremely convoluted.

Return to the field $B(P)$. In four-dimensional Euclidean space, the set of points $[x, y, z, B(x, y, z)]$ is fractal, and it has just been proved by Yoder (1974, first part of the theorem in the appendix) that its D equals $3{\cdot}5$. Thus, in the present case (as in all non-pathological cases of which the author is aware), the fractal dimension exhibits the same behaviour under intersection as a Euclidean dimension: it exceeds the dimension of the iso-sets by one.

2.8. *Proof that the set* $B(P) = 0$ *has a dimension*

In this subsection we shall prove that there exists a constant D such that $\langle N(L,\eta)\rangle \sim (L/\eta)^D$. (This argument will continue to hold in §3.) In the next subsection we shall prove that $D = 2{\cdot}5$.

For the first part, we write $\langle N(L/\eta)\rangle$ as the product of the total number of cells $(L/\eta)^3$ and the conditional probability of a cell being non-empty when it has been assumed that such is the case for the whole test cube. The said probability, to be denoted by $f(L,\eta)$, is obviously a function of L/η because of the self-similarity (scalelessness) of the overall definitions. Let us show that it must in fact be of the form $(L/\eta)^{D-3}$. Indeed, pick two ratios r_1 and r_2 and consider a cell of side $\eta_1 = r_1 L$ and a subcell in it of side $\eta_2 = r_1 r_2 L$. One has

Pr (subcell being non-empty if one knows cube is non-empty)

= Pr (subcell being non-empty if one knows cell is non-empty)

× Pr (cell being non-empty if one knows cube is non-empty).

Hence $f(L, r_1 r_2 L) = f(r_1 L, r_1 r_2 L) f(L, r_1 L)$. By iteration, $f(L, r^n L) = [f(L, rL)]^n$. Since $r^n L = \eta$, this yields

$$\log f(L,\eta) = n \log f(L, rL) = \log(L/\eta) \log f(L, rL)/\log(1/r).$$

Denoting $\log f(L, rL)/\log(1/r)$ by $3 - D$, we prove $f(L,\eta) = (L/\eta)^{3-D}$ for certain L and η; the desired result follows by a well-known interpolatory argument.

2.9. *Outline of a proof that* $D = 2{\cdot}5$

Consider a vertical stack of cells of side η within the large cube, and designate by $N_1(L,\eta)$ the number of those which intersect $B(P) = 0$. It is not hard to believe, and we shall assert it without proof, that the conditional probability distribution of this number, knowing that $N(L,\eta) > 0$, is the same for every stack; so is its probability of being non-zero. Since there are $(L/\eta)^2$ such stacks, we see that $\langle N(L,\eta)\rangle = (L/\eta)^2 \langle N_1(L,\eta)\rangle$, so the result we wish to prove now reads $\langle N_1 L,\eta)\rangle \sim (L/\eta)^{0{\cdot}5}$. Next $N_1(L,\eta)$ is replaced by a variant defined by drawing, through the centres of all the cells of side η in our stack, a 'central line' made of L/η segments of length η and counting the number $N_1^*(L,\eta)$ of those which contain at least one point where $B(P) = 0$. The function $B(P)$ being continuous, we can safely believe that $\langle N_1(L,\eta)\rangle$ differs from $\langle N_1^*(L,\eta)\rangle$ only by a random factor of the order of unity, independent of L and η. Moreover, one can write $\langle N_1^*(L,\eta)\rangle$ as the product of two factors: (*a*) the probability that $N_1 > 0$ and (*b*) the conditioned expectation of N_1^*, to be denoted by $\langle\langle N_1^*(L,\eta)\rangle\rangle$, which is computed by taking account only of the stacks within which N_1^* does not vanish. To evaluate factor (*a*), it suffices to project our surface $B(P) = 0$ onto the 'floor' of the big cube; this projection has a well-defined positive area; factor (*a*) is the ratio of this area to L^2, is non-zero and has the same value for the central lines of all possible stacks. Finally, all that is left to show is that factor (*b*) is proportional to $(L/\eta)^{0.5}$. This task can be reduced to proving that, within a time t after the beginning of a discrete random walk, the expected number of its returns to the point of departure is proportional to $t^{\frac{1}{2}}$. This last fact is well known; see Feller (1968, p. 86, theorem 2).

2.10. *A conjecture*

A stronger result is known for N_1^*, namely that $N_1^*(L,\eta)/\langle N_1^*(L,\eta)\rangle$ is a universal random variable. Doubtless the same is true (but I did not attempt to prove it) of the ratio $N(L,\eta)/\langle N(L,\eta)\rangle$. If this conjecture were confirmed, one could show that the limit $\lim_{\eta\to 0} N(L,\eta)/\log(L/\eta)$ would involve no randomness and could serve as an alternative definition of D. This would bring us closer to the bulk of the mathematical theory of fractal dimensions, and to all earlier applications I made of it, where D was defined either as $D = \log N/\log(L/\eta)$ or as some limit of this expression for $\eta \to 0$. For example, when attempting to show that coastlines are best regarded as curves with a dimension D between 1 and 2, I argued (*a*) that a coastline is self-similar, in the sense that one can, by selecting $N-1$ points on any piece of it, subdivide it into subpieces reduced from the whole by a similarity of ratio r, and (*b*) that, as $r \to 0, \log N/\log r^{-1}$ has a limit that serves to define D. In another application, in which I showed that certain error patterns are best regarded as set with a dimension D below 1, it was proved that the counterpart of $N(L,\eta)/\langle N(L,\eta)\rangle$ is a universal random term.

3. The geometry of random scalar fields with Kolmogorov variance and weighted Poisson and Gaussian distributions: fractional Brownian functions of a point

Since Burgers variance does not hold in fluids, the next approximation to be considered is the Gaussian field $B^*(P)$, with $\langle[B^*(P')-B^*(P'')]\rangle = 0$ and the Kolmogorov variance $\langle[B^*(P)-B^*(P'')]^2\rangle = \overline{|P'P''|}^{\frac{2}{3}}$. The approximations $B(P)$ and $B^*(P)$ are special cases, corresponding respectively to $H = \frac{1}{2}$ and $H = \frac{1}{3}$, of the more general one-parameter family of Gaussian fields $B_H(P)$ such that the variance of the basic increment is $\overline{|P'P''|}^{2H}$, with H a constant between 0 and 1. The case where the space of the P is a line, and $H \neq \frac{1}{2}$, was alluded to by

Kolmogorov (1940) and has been widely used since the work of Mandelbrot & Van Ness (1968), who coined for it the term 'fractional Brownian process'. The case where the space of P is multi-dimensional was briefly alluded to by Yaglom (1957) and by Gangolli (1967), and studied in Mandelbrot (1975*b*).

Fractional Poisson fields: sharp discontinuity. Let us assume everything said about Poisson fields, except that the definition of $D_n F(P)$ will be replaced by $D_n F^*(P) - D_n F^*(O)$, where $D_n F^*(P)$ vanishes if $\overline{OP}.\overline{OH}_n = R_n$ and elsewhere is replaced by

$$D_n F^*(P) = 2^{-1} Q_n \operatorname{sgn}[\overline{OP}.\overline{OH}_n - R_n][\overline{OP}.\overline{OH}_n - R_n]^{-\frac{1}{6}}.$$

If the exponent were 0 instead of $-\frac{1}{6}$, one would, up to an additive factor (due to the fact that it is awkward otherwise to normalize $F^*(O)$ to vanish), go back to $D_n F(P)$. The most striking facts about $F^*(P)$ are that $F^*(P') - F^*(P'')$ tends to infinity near each discontinuity plane and that it is everywhere affected by every one of an infinite number of close or distant discontinuity planes. By way of contrast, $F(P)$ was only affected by the finite number of planes which intersect the segment $\overline{P'P''}$. Thus the elementary steps required for the variance to follow the $\frac{2}{3}$ law are extremely global.

Probability distribution in a scalar weighted probability field. Again, as in §2, the distribution of $F(P)$ tends towards a Gaussian one as $|\overline{OP}| \to \infty$.

Fractional Brownian fields: continuity. As before, the field $\lambda^{-\frac{1}{2}} F^* \lambda(P)$ is defined such that its variance is independent of λ and its $\lambda \to \infty$ limit is a Gaussian field, and hence is identical to $B^*(P)$. One can prove that this limit is continuous (almost surely, almost everywhere), just as was the case for $B(P)$. This property is less obvious here than it was for $B(P)$, because the jumps in $\lambda^{-\frac{1}{2}} D_n F(P)$ tended to zero as $\lambda \to \infty$, while $\lambda^{-\frac{1}{2}} D_n F^*(P)$ remains (for all λ) grossly discontinuous. (Thus the small steps in the limit are obtained as the form $0 \cdot \infty$, rather than $0 \cdot 1$.)

Iso-surfaces. In the fractional case, the Poisson iso-surfaces are less useful than in the Burgers case: first, they are no longer made up of small pieces of plane, and second, the fact that $F^*(P)$ tends to infinity near one side of each plane of discontinuity means that each iso-surface is cut up into small pieces, each of them located within one of the bounded polyhedrons defined by the planes of discontinuity. However, these features cease to be drawbacks in simulations, because $F^*(P)$ is necessarily interpolated from its values computed on a discrete grid. Figure 2 shows several successive 'slices' of the volume enclosed by a sample fractional Brownian iso-surface. It was obtained by interpolating a sample fractional Poisson iso-surface $B^*(P) = 0$, which, again, had been constructed using a Gaussian distribution for the Q_n. Everything is qualitatively as on figure 1, but the irregularity is even more accentuated.

The fractal dimension of the iso-surface is $3 - \frac{1}{3}$, *and that of the surface* $[x, y, z\ B(x, y, z)]$ *is doubtless equal to* $4 - \frac{1}{3}$. All the arguments concerning the fractal dimension run exactly as in §2 until the point where $\langle\langle N_1(L, \eta)\rangle\rangle$ is shown to be proportional to $(L/\eta)^{0 \cdot 5}$. The change is that, for the linear cross-sections, which are fractional Brownian scalar motions of exponent H, this factor equals $(L/\eta)^{1-H}$. For turbulence, we know that $H = \frac{1}{3}$.

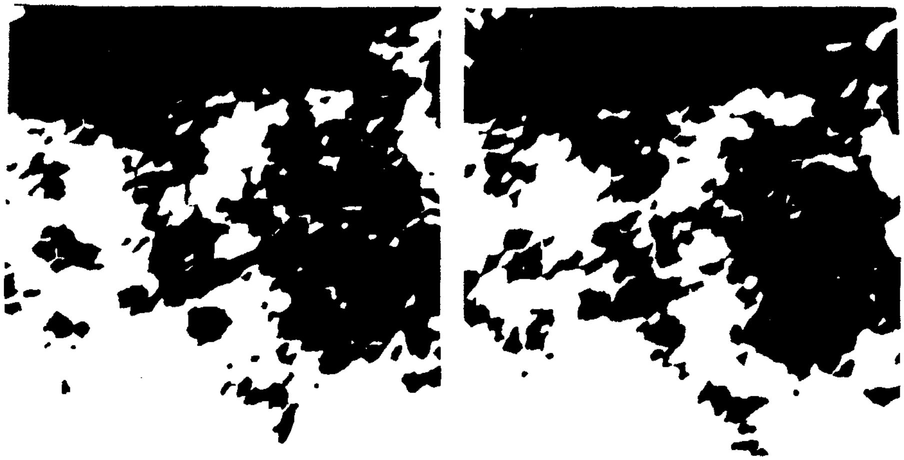

FIGURE 2. Two successive slices of a computer-generated approximation to the volume $B^*(P) > 0$, where $B^*(P)$ is the Gaussian approximation to a Kolmogorov scalar field. Note that (contrary to figure 1) the interpenetrating black and white regions are very much alike in their appearance. The resemblance between this graph and ink splotches (which almost caused it to be destroyed) enhances the feeling that it represents something of the actual motion of fluids.

A self-similar random field that is not Gaussian cannot be approximated by a Poisson field. The increments of a field $F(P)$ are called self-similar if one can rescale them to obtain a function $A(|\overline{OP}|)[F(P)-F(O)]$ having a distribution independent of P. One can show that as a result $A(|\overline{OP}|)$ must be a power of $|\overline{OP}|$, and it follows for example that the degree of flatness (also the kurtosis) of $F(P)$ must be independent of P. Further, one can show that the only self-similar Gaussian fields are those of the form $B_H(P)$, as above, and that the only weighted Poisson fields with a variance of the form $|\overline{OP}|^{2H}$ are again those described above. Hence we see that attempts to obtain a non-Gaussian self-similar field as the limit of weighted Poisson fields are doomed. In §5, the field $F(P)$ will be replaced by a vector field, which can (and in turbulence must) be skew; the warning expressed by this last remark will therefore extend to the impossibility of approximating skew fields using Poisson fields. This result means that the plane discontinuities used above *are mere mathematical devices, devoid of physical meaning.* This feature might have been suspected, given the profoundly nonlinear character of turbulence. The question then arises of whether or not the above results concerning the role and value of fractal dimensions continue to apply to non-Gaussian fields. The answer is not known, and it may well remain so until very specific details of the distribution of the field are determined.

4. Degrees of locality in scalar fields

This section will be devoted to the study of local and global characteristics in Gaussian fields having either Burgers or Kolmogorov variance, or more generally, having the variance $|\overline{P'P''}|^{2H}$ with $0 < H < 1$, called *fractional* if $2H \neq 1$.

It is a good thing to begin with a visual comparison of the figures in Mandelbrot (1975*a*, *c*), which exhibit sample functions of $B_H(x, y, 0)$ for $H = \frac{1}{2}$ and $\frac{1}{3}$. The latter is 'flatter', and its strong high-frequency detail overwhelms a weak low-frequency background.

The field $B(P)$ with Burgers variance. All rectilinear cross-sections of $B(P)$, say $B(x, 0, 0)$, are classical Brownian motions on a line, which are well known to have the Markov property. If one knows $B(0, 0, 0)$, which we may set as equal to zero, then all $B(x, 0, 0)$ for $x < 0$ are independent from all $B(x, 0, 0)$ for $x > 0$; hence the conditional distribution of $B(x_0, 0, 0)$, for a fixed $x_0 > 0$, is the same if one knows $B(x, 0, 0)$ only for $x = 0$, or if, in addition, one knows it for any number of points $x_n < 0$ $(n \geqslant 1)$. Also, if one knows $B(0, 0, 0)$ and $B(x_0, 0, 0)$, added information about the value of $B(x, 0, 0)$ *outside* the interval $(0, x_0)$ does not change the conditional distribution of $B(x, 0, 0)$ *within* that interval. For example, let us extrapolate and interpolate using the expected value. If it is conditioned by $B(0, 0, 0)$, by $B(x_0, 0, 0)$ and by the values of $B(x, 0, 0)$ for any number of points inside $(0, x_0)$, the extrapolate is equal to $B(0, 0, 0)$ for $x < 0$ and to $B(x_0, 0, 0)$ for $x > x_0$. If it is conditioned by $B(0, 0, 0)$, by $B(x_0, 0, 0)$ and by the values of $B(x, 0, 0)$ for any number of points outside $(0, x_0)$, the interpolate is linear. This last feature is a way of expressing that the local behaviour of $B(x, 0, 0)$ in any bounded domain on the line is determined by its values on the domain's boundary, plus additional random effects of local nature.

For the field $B(P)$ in space, the situation turns out to be somewhat different. The local behaviour of $B(x, 0, 0)$ continues to be determined locally, but the meaning of the term 'local' changes substantially. On the one hand, Lévy (1948) showed that, if $B(x, y, 0)$ is known on the horizontal plane, its values above and below that plane are *not* independent; in fact they have a strong negative correlation. Hence knowing the value $B(0, 0, z)$ *does change* the distribution of the value $B(0, 0, -z)$. Similarly, if $B(P)$ is known on a sphere including O, knowing the value $B(0, 0, 0)$ *does change* the distribution of $B(P)$ outside the sphere. On the other hand, McKean (1963) has shown that if $B(P)$ is known on *two* non-intersecting spheres containing O, the distributions of $B(P)$ at two points one inside and the other outside both spheres are indeed independent; this result is called a two-stage Markov property. Thus it remains true that the distribution of $B(P)$ within a small bounded domain is determined locally, but one must define this last term so as to imply a knowledge of $B(P)$ on a double (not a single) boundary.

The field $B^*(P)$ with Kolmogorov variance. Compared with the spectral density k^{-2} of the Burgers field $B(P)$, the spectral density $k^{-\frac{5}{3}}$ of $B^*(P)$ is richer in high-frequency and poorer in low-frequency harmonics, which indicates that it should, if anything, be even more local. However, this issue deserves a more specific examination. We begin again with $B^*(x, 0, 0)$ as an example of a rectilinear cross-section. It is the fractional Brownian random function of one parameter ('time'). It is non-Markovian to an extreme degree: $B^*(0, 0, 0) = 0$ being known, the additional knowledge of any value $B^*(x', 0, 0)\,(x' < 0)$ will affect strongly the distribution of $B(x_0, 0, 0)$, especially when $x_0 = -x'$. *A fortiori*, the same is true of $B^*(P)$ in space. However, curiously enough, this feature is fully compatible with an appropriately weakened concept of what is 'local'.

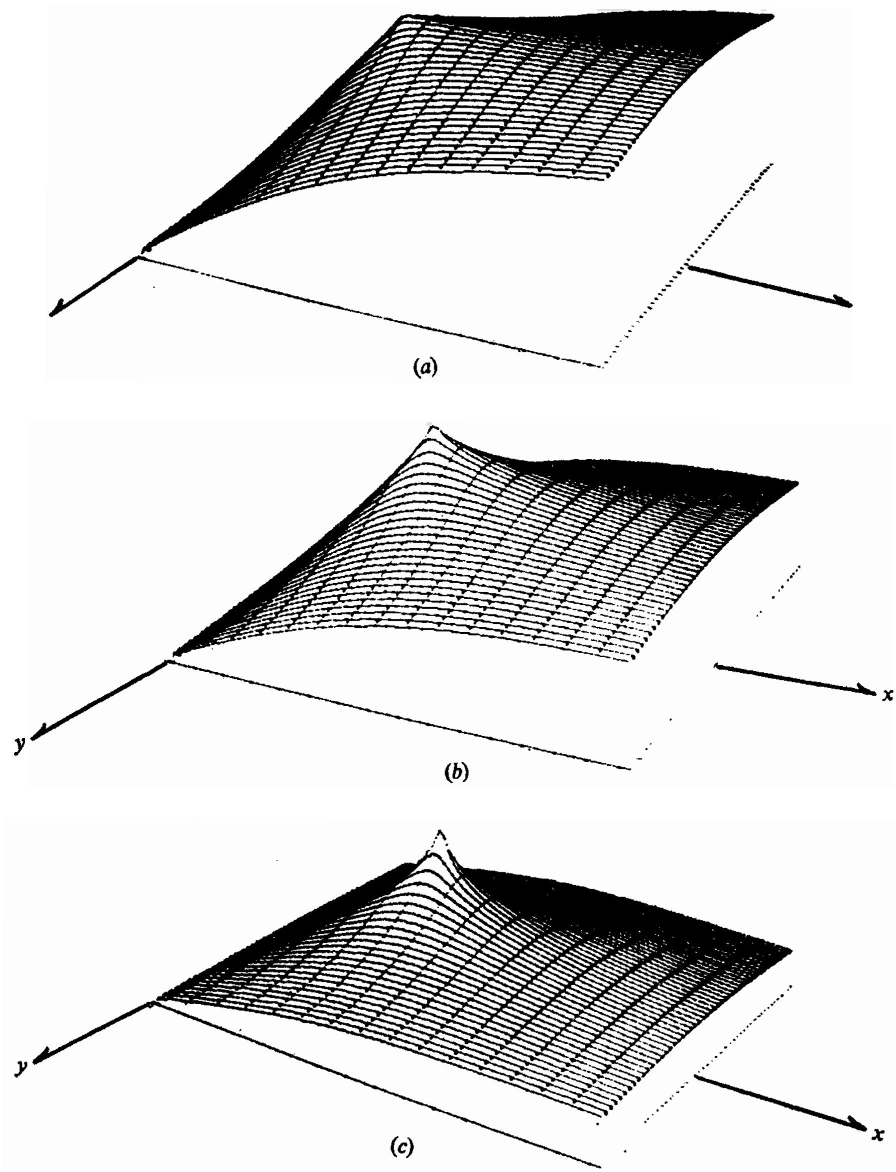

FIGURE 3. The function $Q(x, y, 0)$ for $x > \frac{1}{2}$. The peak's abscissa is $x = 1$. (a) $H = \frac{1}{2}$, (b) $H = \frac{1}{3}$ and (c) $H = \frac{1}{6}$.

The field $B_H(P)$ with $\overline{|P'P''|}^{2H}$ variance. This last characteristic continues to hold true for all Gaussian fields, $B_H(P)$ such that $\langle [B_H(P') - B_H(P'')]^2 \rangle = \overline{|P'P''|}^{2H}$, with $0 < H < \frac{1}{2}$. The overall effect of knowing $B_H(O) = 0$ and $B_H(P_0)$, with $P_0 = (x_0, 0, 0)$, can be assessed by examining the conditional expected value of $B_H(P)$. One finds $\langle [B_H(P) - \frac{1}{2} B_H(P_0)] | B_H(P_0) \rangle = Q B_H(P_0)$, with

$$Q = \tfrac{1}{2}[(\overline{|OP|}/x_0)^{2H} - (\overline{|PP_0|}/x_0)^{2H}].$$

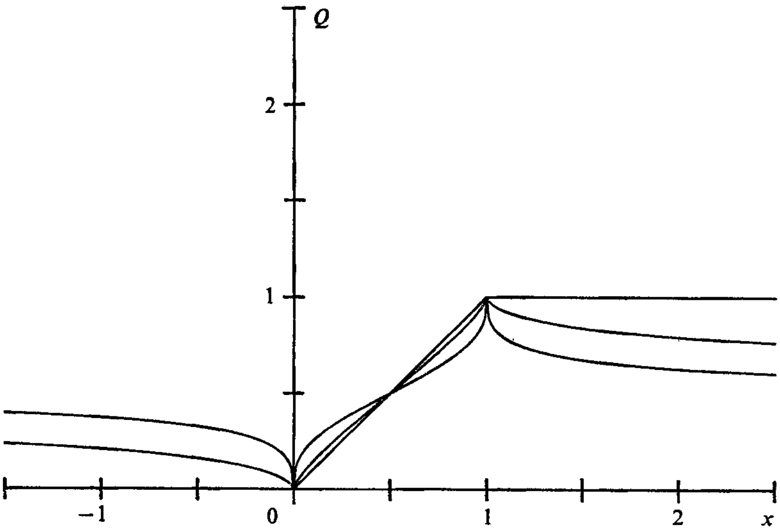

FIGURE 4. The function $Q(x, 0, 0)$, corresponding to P lying on the line joining O and P_0. To the right (or the left) of $x = \frac{1}{2}$, Q is a monotone increasing (or decreasing) function of H. Its behaviour for $H = \frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$ is shown.

To simplify, we shall henceforth set $x_0 = 1$. When P lies on the line from O to P_0, Q reduces to

$$Q = \tfrac{1}{2}[|x|^{2H} - |1-x|^{2H}].$$

We have already described the behaviour of the function Q when H takes the Markov value $H = \frac{1}{2}$ and P lies on the straight line joining O and P_0. Let us extend the examination to other H's and P's. For all H, one has $Q = 0$ along the plane $x = \frac{1}{2}$, $Q > 0$ for $x > \frac{1}{2}$ and $Q < 0$ for $x < \frac{1}{2}$.

More specifically, let $H = \frac{1}{2}$. The iso-Q-surfaces are hyperboloids with rotational symmetry around the x axis. Also, when $x > 1$ and $x^2+y^2+z^2 = r^2 \gg 1$, one has $Q \sim x/r$, meaning that, within a cone around the x axis, Q is close to its upper bound of unity. A perspective view of the surface $Q(x,y,0)$ is shown on figure 3(*a*). Its behaviour shows that, when $|\overline{OP}| \gg |\overline{OP_0}|$, the conditioned distribution of $B(P)$, knowing $B(O)$ and $B(P_0)$, depends strongly upon the precise relative positions of P, O and P_0. Conversely, in some cases, as when $x \gg 1$ and $y = z = 0$, $B(P)$ affects the distribution of $B(P_0)$ but not at all the distribution of $B(O) - B(P_0)$; in other cases, as when $r \gg 1$, and x is near zero, $B(P)$ affects the distribution of $\frac{1}{2}[B(P_0)+B(O)]$ but hardly at all the distribution of $B(O) - B(P_0)$.

Now let $H < \frac{1}{2}$. Two perspective views of the surface $Q(x,y,0)$ and a precise graph for $y = z = 0$ are shown on figures 3(*b*), 3(*c*) and 4. On the latter, for $x = \frac{1}{2}$, $Q = \frac{1}{2}$ and has a slope equal to $H \times 2^{1-2H}$ (for $H = \frac{1}{3}$, it is equal to 0·84); at the points $x = 0$ and $x = 1$, it has cusps, but over most of $(0, 1)$, it is not too far from being straight. When P is on the x axis outside $(0, 1)$, Q tends to zero. The same is true going away from O along other directions, since it is readily seen that, when $r^2 = x^2+y^2+z^2 \gg 1$, $Q_H \sim (x/r)\, r^{2H-1}$. Thus, for $H \neq \frac{1}{2}$ and $r \gg 1$, the conditioned distribution of $B_H(P)$, knowing $B_H(O)$ and $B_H(P_0)$, depends primarily on $\frac{1}{2}[B_H(O)+B_H(P_0)]$; knowledge of the precise relative positions of O, P and P_0 leads only to a second-order correction. Conversely, $B_H(P)$ mostly affects the local average $\frac{1}{2}[B_H(O)+B_H(P_0)]$ and hardly at all the distribution of $B_H(O)-B_H(P_0)$.

In summary, neither $B(P)$ nor $B_H(P)$ is strictly local for $H<\frac{1}{2}$. But from certain viewpoints, locality is more marked for $B_H(P)$ than for $B(P)$. This is already true when $H=\frac{1}{3}$. If H were below the Kolmogorov value $\frac{1}{3}$, this locality would be even more accentuated.

The more local behaviour of Q near O and P_0 is also worth dwelling upon, if only to confirm it is indeed 'local'. This is a region where the expected behaviour of $B_H(P)$ is even more important in comparison with the random component. We see that the local shape is likely to consist of a sharp pit located next to a sharp peak, where 'sharp' means extending away to a few times x_0. This feature helps to explain the presence of small pieces of 'jetsam and flotsam' observed earlier in the paper.

5. The geometry of some random vector fields

We shall sketch in few words the generalization of the results in §§2 and 3 to vector fields with Burgers variance. They may, again, be of assistance in building up intuition, but their counterpart for the case of Kolmogorov variance is likely to have very limited use, because skewness is of the essence and there is no natural way of building it in using the approach in this paper. For example, one can show that the probability distribution in a weighted Poisson field is such that, while $\langle[F(P)-F(O)]^2\rangle = |\overline{OP}|^{2H}$, $\langle[F(P)-F(O)]^3\rangle$ equals $|\overline{OP}|^{3H-0\cdot5}$ if $H>\frac{1}{6}$ and is roughly constant if $H<\frac{1}{6}$. Hence, the skewness of $F(P)-F(O)$ unavoidably tends to zero as $|\overline{OP}|$ increases.

Brownian vector fields. A Brownian vector field (normalized, for the sake of convenience, to satisfy $\mathbf{B}(O)=0$) can be defined naturally as a vector $\mathbf{B}(P)$ such that its normal component and its two tangential components are independent Gaussian random variables of variance proportional to $|\overline{OP}|$. It is easy to see that the normal component is at most as large as either of the tangential ones; this follows from the Kármán–Howarth (1938) reduction of a correlation term to the functions they call $f(r,t)$ and $g(r,t)$.

Poisson vector fields. Similarly, one can generate a Poisson field such that its planar discontinuities have both a normal and a tangential component.

Brownian limits of Poisson vector fields. One may construct a Brownian field $\mathbf{B}(P)$ as a limit $\lim_{\lambda\to\infty}\mathbf{F}_\lambda(P)$ of a sequence of Poisson fields; however, this can only be a special Brownian field, because it is readily seen that in the resulting field the vector $\mathbf{B}(P)-\mathbf{B}(O)$ is isotropic at every point; in the Kármán–Howarth notation, the functions $f(x,t)$ and $g(x,t)$ are identical.

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